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Limit Theorems for Random Point Measures Generated by Cooperative Sequential Adsorption

Vadim Shcherbakov[1]

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We consider a finite sequence of random points in a finite domain of finite-dimensional Euclidean space. The points are sequentially allocated in the domain according to the model of cooperative sequential adsorption. The main peculiarity of the model is that the probability distribution of any point depends on previously allocated points. We assume that the dependence vanishes as the concentration of points tends to infinity. Under this assumption the law of large numbers, Poisson approximation and the central limit theorem are proved for the generated sequence of random point measures.

KEY WORDS: cooperative sequential adsorption with infinite range cooperative effects, the law of large numbers, Poisson approximation, the central limit theorem, Gaussian random field

1. INTRODUCTION AND THE RESULTS

In this paper we study the asymptotic behavior of random point measures

$$\mu_m = \sum_{i=1}^{m} \delta_{X_i}, \tag{1}$$

generated by random points $X_1, \ldots, X_m$ sequentially allocated in a compact set $D \subset \mathbf{R}^d$. To describe the joint distribution of $X_1, \ldots, X_m$ we need some notation. For any point $x \in D$ and a finite non-empty set $\mathbf{y} = \{y_1, \ldots, y_n\},\ y_i \in D, n \geq 1$, we denote by $n(x, \mathbf{y})$ the number of points $y_i \in \mathbf{y}$, such that the distance between x and y_i is not greater than $R(x)$, where $R : D \to \mathbf{R}_+$ is some measurable function. By definition $n(x, \emptyset) = 0$. The number $R(x)$ is called the interaction radius at point x. Let $\{\beta_n(x),\ n \geq 0\}$ be a sequence of measurable positive bounded functions on

[1] Laboratory of Large Random Systems, Faculty of Mechanics and Mathematics, Moscow State University 119992, Moscow, Russia; e-mail: Shcherbakov@mech.math.msu.su

D. Denote for short $X(k) = (X_1, \ldots, X_k),\ k \geq 1$, and $X(0) = \emptyset$. Given the set of points $X(k)$ the conditional distribution of point X_{k+1} is specified by the following probability density

$$\psi(x \mid X(k)) = \frac{\beta_{n(x,X(k))}(x)}{\alpha(X(k))}, \tag{2}$$

where

$$\alpha(X(k)) = \int_D \beta_{n(u,X(k))}(u)\, du,$$

is the normalizing constant. The joint probability density of $X_1, \ldots, X_m$ at points $x_1, \ldots, x_m$ is

$$p_m(x_1, \ldots, x_m) = \prod_{k=1}^{m} \frac{\beta_{n(x_k,\mathbf{x}_{<k})}(x_k)}{\int_D \beta_{n(x,\mathbf{x}_{<k})}(x)dx} = \prod_{k=1}^{m} \psi(x_k \mid \mathbf{x}_{<k}), \tag{3}$$

where we denoted for short $\mathbf{x}_{<k} = (x_1, \ldots, x_{k-1}),\ k \geq 2$, and $\mathbf{x}_{<1} = \emptyset$ for $k = 1$.

Let us give examples of the situations where this set of sequentially allocated random points naturally appears. First we do it in terms of continuous time dynamic processes describing adsorption reactions with cooperative effects. Namely, consider a spatial birth process $\mathbf{x}(t),\ t \geq 0$, in D with birth rates defined in terms of functions $\beta_n(x),\ n \geq 0$, as follows. If the process state at time $t \geq 0$ is $\mathbf{x}$, then the birth rates are $\beta_{n(x,\mathbf{x})}(x),\ x \in D$, so the total birth rate is $\alpha(\mathbf{x})$ and the time until the next jump is an exponential random variable with mean $\alpha^{-1}(\mathbf{x})$. Assume that $\mathbf{x}(0) = \emptyset$ and consider a random point process $X(m) = (X_k,\ k = 1, \ldots, m)$ formed by the first m points of the spatial birth process $\mathbf{x}(t)$. It is easy to see that the first point X_1 has the probability distribution specified by the function $\beta_0(x)$ normalized to be a probability density. Given $X_1, \ldots, X_k,\ k \geq 1$, the conditional distribution of X_{k+1} is specified by the probability density (2). The spatial birth process just described is a continuous version of a lattice model of monomer filling with nearest-neighbor cooperative effects. It is a particular case of the models of cooperative sequential adsorption widely used in physics and chemistry for modeling various adsorption processes (see Refs. 2 and 5 for more details and surveys of the relevant literature).

The set of random points $X(m)$ can be also viewed as the output of the following sequential packing process. Consider random points $Y_i,\ i \geq 1$, sequentially arriving in D. Each point Y_i is uniformly distributed in D and is accepted with probability depending on the number of previously accepted points in the local configuration near Y_i. More precisely, let $Y(N) = (Y_1, \ldots, Y_N)$ be a set of the first N arrived points and let $X(k) = (X_1, \ldots, X_k),\ k = k(N)$, be a set of accepted ones among $Y_i,\ i = 1, \ldots, N$. Next uniformly distributed arrival Y_{N+1} is accepted with probability $\beta_{n(Y_{N+1},X(k))}(Y_{N+1})/C$, where C is an arbitrary constant such that

$\sup_n \sup_{x\in D} \beta_n(x) \leq C$. Regardless of the particular choice of C the probability density of the next *accepted* point X_{k+1} is given by the formula (2). The value of C influences only the number of discarded arrivals until next acceptance. In other words, given the set of previously accepted points $X(k)$, we use a well known acceptance-rejection sampling for simulating a random variable which distribution is specified by the unnormalized probability density $\beta_{n(x,X(k))}(x)$, $x \in D$. The sequence of points $X(m)$ is a set of first m sequentially accepted points.

The measures (1) belong to the class of random point measures generated by the spatial processes arising in random sequential packing and deposition problems (see Refs. 1, 6 and references therein). The typical example is when one sequentially allocates m points in the unit cube. Each point is uniformly distributed in the cube and is accepted with probability depending on configuration of previously accepted points in a ball of volume $1/m$ centered at the point. It means that the interaction radius decays and a point typically has a finite number of neighbors in the limit $m \to \infty$. This regime corresponds to finite range interaction between points. It is not the case in our model where the interaction radius is a fixed positive function (or constant) regardless of the number of points. In our case the radius is comparable to the volume size (it implies that a typical number of neighbors is linear in m). In some sense this corresponds to the so-called infinite range of interaction or infinite range cooperative effects, see, for instance, Ref. 2.

Our other main assumption is that $\beta_n(x) \to \beta(x) > 0$ as $n \to \infty$ uniformly in $x \in D$, where function β is bounded below and above. Under our assumptions the sequence of random variables X_k, $k \geq 1$, converges in total variation to a random variable with the probability density specified by function $\beta(x)$, $x \in D$, appropriately normalized. Therefore the model can be considered as a perturbation of the binomial case which is $\beta_n(x) = \beta(x)$, $x \in D$ for any $n \geq 0$. The perturbation vanishes while the domain is saturated by points. The distribution of a new arrival becomes "more uniform" and "more independent" on the existing configuration of points provided the domain is sufficiently saturated and the saturation is "sufficiently uniform." We make it rigorous in Lemma 1.1. In the binomial case we immediately get Theorems 1.1, 1.2 and 1.3, since the points are independent. In general case the points are dependent and we arrive at the proof of the law of large numbers, the central limit theorem and Poisson approximation for the sequence of *dependent* random variables. Some care should be taken to assess the weakening of dependence in the tail of the sequence $X(m)$. Note that we obtain the central limit theorem (Theorem 1.3) assuming that the sequence of functions $\{\beta_n(x), n \geq 0\}$ converges to its limit with some rate.

These limit theorems can be used for understanding the qualitative behavior of high density point patterns obtained by an idealized adsorption process specified by two conditions. The first one is that the interaction radius is sufficiently large. This is modeled by assuming that the radius does not decrease as the number of points increases. The second one is that a point is allowed to have unlimited number of neighbors. This is provided by soft-core type of interaction, i.e., the reaction rates are positive regardless of configuration (positiveness of functions β's). Besides, the reaction rates depend on local environment and stabilize when the concentration of adsorbed molecules is sufficiently high.

Remark 1. We will denote by the letter C or by the letter C with subscripts the various constants the particular values of which are immaterial for the proofs. By $\mathcal{B}(D)$ the set of real-valued measurable bounded functions on D is denoted and $\|f\|_\infty = \sup_{x\in D} |f(x)|$ for $f \in \mathcal{B}(D)$. It is assumed that the random variables X_k, $k \geq 1$, are realized on some probability space with probability measure P and E is expectation with respect to P.

Theorem 1.1. *Assume that* $\inf_{x\in D} R(x) > 0$*, the sequence of positive functions* $\beta_n \in \mathcal{B}(D)$, $n \geq 0$, *is uniformly bounded and converges uniformly as* $n \to \infty$ *to a function* $\beta \in \mathcal{B}(D)$*, such that* $\inf_{x\in D} \beta(x) > 0$. *Then the law of large numbers holds for the sequence of random measures* μ_m. *That is for any function* $f \in \mathcal{B}(D)$

$$\frac{1}{m}\int_D f(x)d\mu_m(x) = \frac{1}{m}\sum_{i=1}^{m} f(X_i) \to J(f) = \frac{1}{\alpha}\int_D f(x)\beta(x)\,dx,$$

in probability as $m \to \infty$*, where* $\alpha = \int_D \beta(x)\,dx$.

Theorem 1.2. *In addition to the assumptions of Theorem 1.1 assume that the function* β *is continuous. Fix an arbitrary* $x \in D$ *and* $r > 0$. *Let* $S_m(x,r)$ *be a number of those points* X_k, $k = 1, \ldots, m$, *that fall in a ball* $B(x, rm^{-1/d})$. *Then a sequence of random variables* $S_m(x,r)$, $m \geq 1$, *converges in distribution to a Poisson random variable with parameter* $r^d b_d \beta(x)/\alpha$*, where* b_d *is a volume of a* $d-$*dimensional ball with unit radius.*

Theorem 1.3. *In addition to the assumptions of Theorem 1.1 assume that*

$$|\beta_n(x) - \beta(x)| \leq \tau(x)\varphi(n), \tag{4}$$

for any $n \geq 0$*, where a function* $\varphi(s) > 0$, $s \geq 0$, *is such that* $\varphi(s) \to 0$ *as* $s \to \infty$ *and for any* $\delta > 0$

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n} \varphi(k\delta) \to 0, \tag{5}$$

as $n \to \infty$*, the function* $\tau \in \mathcal{B}(D)$ *is such that* $\inf_{x\in D} \tau(x) > 0$. *Then the sequence centered and rescaled random measures* $(\mu_m - \mathsf{E}\mu_m)/\sqrt{m}$ *converges as* $m \to \infty$ *to a generalized Gaussian random field on* D *with zero mean and the covariance* *kernel*

$$G(f, g) = J(fg) - J(f)J(g)$$
$$= \frac{1}{\alpha}\int_D f(x)g(x)\beta(x)\,dx - \frac{1}{\alpha^2}\int_D f(x)\beta(x)\,dx \int_D g(x)\beta(x)\,dx,$$

for any functions $f, g \in \mathcal{B}(D)$*. Here the convergence refers to the convergence of the corresponding sequence of finite-dimensional distributions.*

To prove these theorems we will use Lemmas 1.1–1.4.

Lemma 1.1. *Assume that* $\inf_{x\in D} R(x) > 0$ *and*

$$0 < \beta_{\min} = \inf_n \inf_{x\in D} \beta_n(x) \le \beta_{\max} = \sup_n \sup_{x\in D} \beta_n(x) < \infty,$$

then there exists a positive constant δ_0 *such that for any* $\delta \in (0, \delta_0)$

$$\mathsf{P}\Big\{\inf_{x\in D} n(x, X(m)) \le m\delta\Big\} \le Ce^{-\lambda m}, \tag{6}$$

with some positive constants $C = C(\delta)$ *and* $\lambda = \lambda(\delta)$ *for all sufficiently large* m*. If the assumptions of Theorem 1.1 hold, then for any* $\varepsilon > 0$

$$\mathsf{P}\Big\{\sup_{x\in D} |\beta_{n(x,X(m))}(x) - \beta(x)| \ge \varepsilon\Big\} \le Ce^{-\lambda m}, \tag{7}$$

and

$$\mathsf{P}\{|\alpha(X(m)) - \alpha| \ge \varepsilon\} \le Ce^{-\lambda m} \tag{8}$$

with the same positive constants C *and* λ *for all sufficiently large* m*.*

Corollary 1.1. *If the assumptions of Theorem 1.1 hold, then the sequence* X_m*,* $m \ge 1$*, converges in total variation to a random variable* X *distributed according to the density* $\beta(x)/\alpha$*, as* $m \to \infty$*.*

Let $\mathcal{F}_{k-1}$ be a $\sigma-$algebra generated by the random variables $X_1, \ldots, X_{k-1}$. For any function $f \in \mathcal{B}(D)$ denote

$$J_k(f) = \mathsf{E}(f(X_k)\,|\,\mathcal{F}_{k-1}).$$

Lemma 1.2.

1) If the assumptions of Theorem 1.1 hold, then for any function $f \in \mathcal{B}(D)$ *and for any* $p \ge 1$

$$\mathsf{E}|J_k(f) - J(f)|^p \to 0$$

as $k \to \infty$*.*

2) If the assumptions of Theorem 1.3 hold and δ_0 is the constant determined in Lemma 1.1, then for any $\delta \in (0, \delta_0)$

$$\mathsf{E}|J_k(f) - J(f)|^p \le C(\varphi^p(k\delta) + e^{-\lambda k})$$

as $k \to \infty$, with some constant $\lambda = \lambda(\delta)$.

Let Y be a random variable with probability density $\beta(x)/\alpha$. For any function $f \in \mathcal{B}(D)$ and $n \ge 1$ denote

$$\mathcal{U}_n(f) = \mathsf{E}(f(Y) - \mathsf{E} f(Y))^n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} J(f^i) J^{n-i}(f), \tag{9}$$

and $\xi_k(f) = f(X_k) - \mathsf{E} f(X_k)$.

Corollary 1.2. *Let $f \in \mathcal{B}(D)$ and fix some positive integer n. Then*
1) under assumptions of Theorem 1.1

$$\mathsf{E}\left|\mathsf{E}\left(\xi_k^n(f)|\mathcal{F}_{k-1}\right) - \mathcal{U}_n(f)\right| \to 0$$

as $k \to \infty$, and
2) under assumptions of Theorem 1.3

$$\mathsf{E}\left|\mathsf{E}\left(\xi_k^n(f)|\mathcal{F}_{k-1}\right) - \mathcal{U}_n(f)\right| \le Cxs(\varphi(k\delta) + e^{-\lambda k})$$

as $k \to \infty$.

Lemma 1.3. *Fix a set of functions $g_1, \ldots, g_k \in \mathcal{B}(D)$ and a set of positive integers $r_1, \ldots, r_k$ and let $n = r_1 + \cdots + r_k$. Let a set of indices be such that $i_1 < \cdots < i_k$ and denote by η a random variable measurable with respect to the $\sigma-$algebra $\mathcal{F}_{i_1-1}$.*
1) If the assumptions of Theorem 1.1 hold, then

$$\left|\mathsf{E}\left(\eta \prod_{v=1}^{k} \xi_{i_v}^{r_k}(g_v)\right) - \mathsf{E}\eta\left(\prod_{v=1}^{k} \mathcal{U}_{r_k}(g_v)\right)\right| \to 0$$

as $i_1 \to \infty$. In particular, for any $f, g \in \mathcal{B}(D)$ and $k \ne j$

$$Cov(f(X_k), g(X_j)) \to 0,$$

as $\max(k, j) \to \infty$.
2) If the assumptions of Theorem 1.3 hold, then there exist constants $C = C(k, g_1, \ldots, g_k)$ such that for any $\delta \in (0, \delta_0)$

$$\left|\mathsf{E}\eta \prod_{v=1}^{k} \xi_{i_v}^{r_k}(g_v) - \mathsf{E}\eta \prod_{v=1}^{k} \mathcal{U}_{r_k}(g_v)\right| \le C^n \sum_{v=1}^{k} \left(\varphi(i_v\delta) + e^{-\lambda i_v}\right), \tag{10}$$

for all sufficiently large indices $i_1 < \cdots < i_k, k \geq 1$, *where the constants* λ *and* δ_0 *are determined in Lemma 1.1.*

Lemma 1.4. *Under the assumptions of Theorem 1.3 the sequence of random variables*

$$\frac{1}{\sqrt{m}} \sum_{k=1}^{m} (J_k(f) - \mathsf{E} f(X_k))$$

converges to 0 *in probability as* $m \to \infty$.

2. PROOFS

Proof of Theorem 1.1. Let us prove first that for any function $f \in \mathcal{B}(D)$

$$\frac{1}{m} \sum_{k=1}^{m} \mathsf{E} f(X_k) \to J(f), \tag{11}$$

as $m \to \infty$. Indeed, By Lemma 1.2 we have that $\mathsf{E} f(X_k) \to J(f)$, as $k \to \infty$. Fix an arbitrary $\varepsilon > 0$ and let $k(\varepsilon)$ be such that $|\mathsf{E} f(X_k) - J(f)| \leq \varepsilon$ as $k > k(\varepsilon)$. It is easy to see that

$$\left| \frac{1}{m} \sum_{k=1}^{m} \mathsf{E} f(X_k) - J(f) \right| \leq 2 \frac{k(\varepsilon)}{m} \|f\|_\infty + \frac{m - k(\varepsilon)}{m} \varepsilon.$$

The first term in the right side of the preceding equation goes to 0 as $m \to \infty$, the second is less than ε. Thus we get (11) since ε is arbitrary. It suffices now to prove that

$$\frac{1}{m} \sum_{k=1}^{m} (f(X_k) - \mathsf{E} f(X_k)) \to 0,$$

in probability as $m \to \infty$. By Chebyshev inequality we have that for any $\varepsilon > 0$

$$\mathsf{P} \left\{ \left| \sum_{k=1}^{m} f(X_k) - \mathsf{E} f(X_k) \right| \geq \varepsilon m \right\} \leq \frac{1}{\varepsilon^2 m^2} \sum_{k,j=1}^{m} Cov(f(X_k), f(X_j)).$$

If $k \neq j$, then by part 1) of Lemma 1.3 $Cov(f(X_k), f(X_j)) \to 0$ as $\max(k, j) \to 0$, therefore the right hand side of the preceding display vanishes as $m \to \infty$. Theorem 1.1 is proved. □

Proof of Theorem 1.2. Let $x \in D$ and $r > 0$ be fixed. Denote for short $S_m = S_m(x, r)$. We prove that for any $t \in \mathbf{R}$

$$\lim_{m \to \infty} \mathsf{E} e^{itS_m} = \exp\{(e^{it} - 1)\beta(x) r^d b_d / \alpha\}. \tag{12}$$

By definition

$$S_m = \sum_{k=1}^{m} \xi_{m,k},$$

where $\xi_{m,k} = 1_{\{X_k \in B(x, rm^{-1/d})\}}$. For any $k \geq 1$ we can write

$$\mathsf{E}\left(e^{it\xi_{m,k}} \mid \mathcal{F}_{k-1}\right) = 1 + (e^{it} - 1)p_m + (e^{it} - 1)(p_{m,k} - p_m), \tag{13}$$

where $p_{m,k} = \mathsf{P}\{X_k \in B(x, rm^{-1/d}) \mid \mathcal{F}_{k-1}\}$ and p_m is the probability that a random variable with density $\beta(y)/\alpha$, $y \in D$, falls in the ball $B(x, rm^{-1/d})$. Repeatedly using the Eq. (13) we obtain that

$$\begin{aligned}\mathsf{E}e^{itS_m} = &\ (1 + (e^{it} - 1)p_m)^m \\ &+ (e^{it} - 1)\sum_{k=1}^{m}(1 + (e^{it} - 1)p_m)^{m-k}(\mathsf{E}p_{m,k} - p_m).\end{aligned}$$

It is easy to see that $mp_m \to \beta(x)r^d b_d/\alpha$ as $m \to \infty$. Therefore the first term in the left hand side of the preceding display tends to the characteristic function of the Poisson distribution with parameter $\beta(x)r^d b_d/\alpha$. Let us show that the second term in the left hand side of the preceding display vanishes as $m \to \infty$. Noting that $p_m = J(f_m)$ and $p_{m,k} = J_k(f_m)$ with function $f_m(y) = 1_{\{y \in B(x, rm^{-1/d})\}}$ and using Remark 1 after the proof of Lemma 1.2 (the bound (29)) we can write

$$\mathsf{E}|p_{m,k} - p_m| \leq \frac{C}{m}\mathsf{E}\sup_{y \in D}\left|\beta_{n(y, X(k-1))}(y) - \beta(y)\right|. \tag{14}$$

Fix an arbitrary $\varepsilon > 0$. An argument leading to the bounds (26) and (27) in the proof of Lemma 1.2 gives us here that there exists such $k(\varepsilon)$ that for any $k \geq k(\varepsilon)$ we can replace the bound (14) by the following one

$$\mathsf{E}|p_{m,k} - p_m| \leq \frac{C_1}{m}(\varepsilon + e^{-\lambda k}), \tag{15}$$

where constant λ is the same as in Lemma 1.1. Hence we can bound

$$\left|(e^{it} - 1)\sum_{k=1}^{m}(1 + (e^{it} - 1)p_m)^{m-k}(\mathsf{E}p_{m,k} - p_m)\right| \leq \left(C_2\varepsilon + \frac{C_3}{m}\right).$$

Therefore we finished the proof since ε was taken arbitrary. □

Remark 2. Using Theorem 1 in Ref. 7 (a general result on Poisson approximation for sums of possibly dependent nonnegative integer-valued random variables) one can also bound

$$\sup_{A \subset \mathbf{Z}_+} |\mathsf{P}\{S_m \in A\} - \mathsf{P}\{Y_m \in A\}| \leq \sum_{k=1}^{m} p_m^2 + \sum_{k=1}^{m} \mathsf{E}|p_{m,k} - p_m|, \tag{16}$$

where Y_m is a Poisson random variable with parameter mp_m. Combining the bound (15) with the fact that mp_m has a finite limit as $m \to \infty$ one can show that the right hand side of the Eq. (16) vanishes as $m \to \infty$.

Proof of Theorem 1.3. It suffices to prove that for any function $f \in \mathcal{B}(D)$ the sequence of random variables

$$S_m(f) = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} (f(X_k) - \mathsf{E} f(X_k)) \tag{17}$$

converges weakly as $m \to \infty$ to a Gaussian random variable with mean zero and the variance $G(f, f) = J(f^2) - J^2(f)$. Note that

$$S_m(f) = Z_m(f) + \frac{1}{\sqrt{m}} \sum_{k=1}^{m} (J_k(f) - \mathsf{E} f(X_k)), \quad m \geq 1, \tag{18}$$

where

$$\begin{aligned} Z_m(f) &= \frac{1}{\sqrt{m}} \sum_{k=1}^{m} (f(X_k) - \mathsf{E}(f(X_k)|\mathcal{F}_{k-1})) \\ &= \frac{1}{\sqrt{m}} \sum_{k=1}^{m} (f(X_k) - J_k(f)), \quad m \geq 1. \end{aligned}$$

By Lemma 1.4 the second term in the right hand side of the Eq. (18) converges to 0 as $m \to \infty$. Therefore to prove the theorem we need to prove that the sequence of random variables $Z_m(f)$, $m \geq 1$, converges weakly to a Gaussian random variable with mean zero and the variance $G(f, f)$ as $m \to \infty$. Note that $\{Z_m(f), \mathcal{F}_m, m \geq 1\}$ is a zero-mean, square-integrable martingale array with differences $\zeta_{mk} = (f(X_k) - J_k(f))/\sqrt{m}$, $k = 1, \ldots, m$. It is easy to see that

$$\max_k |\zeta_{mk}| \leq \frac{2\|f\|_\infty}{\sqrt{m}} \to 0, \tag{19}$$

and

$$\mathsf{E}\left(\max_k \zeta_{mk}^2\right) \leq \frac{4\|f\|_\infty^2}{m} \to 0. \tag{20}$$

By Corollary 1.1 and Lemma 1.2 $\mathsf{E}(f(X_k) - J_k(f))^2$ converges to $G(f, f)$ as $k \to \infty$. Consequently $\sum_{k=1}^{m} \mathsf{E}\zeta_{mk}^2$ converges to $G(f, f)$ as $m \to \infty$. Combining the results of Lemmas 1.2 and 1.3 it is easy to obtain that $Cov((f(X_k) - J_k(f))^2, (f(X_j) - J_j(f))^2$ tends to 0 for $k \neq j$ as $\max(k, j) \to$

∞. It yields that $Var\left(\sum_{k=1}^{m}\zeta_{mk}^2\right)$ vanishes as $m \to \infty$. Therefore

$$\sum_{k=1}^{m}\zeta_{mk}^2 \to G(f, f), \tag{21}$$

in probability as $m \to \infty$.

The Eqs. (19), (20) and (21) mean that the conditions of Theorem 3.2 in Ref. 3 hold for the martingale array $\{Z_m(f), \mathcal{F}_m, m \geq 1\}$. Therefore $Z_m(f)$ converges in distribution to a Gaussian random variable with zero mean and covariance $G(f, f)$ as $m \to \infty$ and Theorem 1.3 is proved.

Proof of Lemma 1.1. Without loss of generality we assume that the set D is a $d-$dimensional unit cube. If $l \in \mathbf{Z}_+$ is the minimal integer such that

$$p(l) = l^{-d}\frac{\beta_{\min}}{\beta_{\max}} < 1, \quad \text{and} \quad 1/l < \frac{1}{4}\inf_{x\in D} R(x),$$

then we put $\delta_0 = p(l)$. Let $\{Q_i, i = 1, \ldots, l^d\}$ be a set of non-overlapping cubes of size $1/l$ such that $D = \bigcup_i Q_i$. Denote by ξ_{mi} a number of points $X_1, \ldots, X_m$ falling in the cube Q_i. Take a point $x \in D$ and let $x \in Q_i$ for some i. It is easy to see that

$$n(x, X(m)) \geq \xi_{mi} \geq \min_j \xi_{mj}, \tag{22}$$

since $Q_i \subset B(x, R(x))$. The Eq. (22) implies that

$$\{n(x, X(m)) \leq z\} \subset A_m = \left\{\min_i \xi_{mi} \leq z\right\} = \bigcup_i \{\xi_{mi} \leq z\},$$

for any $z > 0$. It is obvious that

$$\mathsf{P}\{A_m\} \leq l^d \max_i \mathsf{P}\{\xi_{mi} \leq z\}.$$

The formula (2) yields that

$$\mathsf{P}\{X_k \in Q_i \mid X(k-1)\} = \frac{\int_{Q_i}\beta_{n(u,X(k-1))}(u)du}{\int_D \beta_{n(u,X(k-1))}(u)du}.$$

This conditional probability can be bounded below by $p(l)$ uniformly in sequences $X(k-1)$. Therefore the unconditional probability $\mathsf{P}\{X_k \in Q_i\}$ is also bounded below by the same constant for any $k \geq 1$. Using the well-known coupling construction we can construct on the same probability space the random variable ξ_{mi} and the binomial random variable $\tilde{\xi}_{mi}$ with m trials and with $p(l)$ the probability of success such that ξ_{mi} stochastically dominates $\tilde{\xi}_{mi}$. So, we have that

$$\mathsf{P}\{\xi_{mi} \leq m\delta\} \leq \mathsf{P}\{\tilde{\xi}_{mi} \leq m\delta\}$$

for any $\delta > 0$. If we take δ such that $0 < \delta < \delta_0 = p(l)$, then the well known large deviations bounds for the sums of i.i.d. random variables give us that

$$\mathsf{P}\{\tilde{\xi}_{mi} \le m\delta\} \le Ce^{-\lambda m},$$

with some positive constants C and λ. Therefore

$$\mathsf{P}\Big\{\inf_{x\in D} n(x, X(m)) \le m\delta\Big\} \le l^d \max_i \mathsf{P}\{\xi_{mi} \le m\delta\} \le Cl^d e^{-\lambda m}$$

and the proof of the bound (6) is over. The bounds (7) and (8) are immediate implication of the bound (6) and the convergence of the β's. Indeed, for any $\varepsilon > 0$ we have that $\sup_{x\in D} \mid \beta_{n(x,X(m))}(x) - \beta(x)| < \varepsilon$ as soon as $\inf_{x\in D} n(x, X(m)) > n(\varepsilon)$, for some $n(\varepsilon)$. Lemma 1.1 is proved. □

Proof of Corollary 1.1. By the Eq. (2) the unconditional density of the random variable X_{k+1} at point x is

$$\mathsf{E}\psi(x \mid X(k)) = \mathsf{E}\frac{\beta_{n(x,X(k))}(x)}{\alpha(X(k))}.$$

The integrand in this mean is bounded and converges in probability to $\beta(x)/\alpha$ as $k \to \infty$ by Lemma 1.2. Therefore, $\mathsf{E}\psi(x|X(k)) \to \beta(x)/\alpha$ for any $x \in D$ as $k \to \infty$. It is well known that the point-wise convergence of densities implies the convergence in total variation. Corollary 1.1 is proved. □

Proof of Lemma 1.2. To simplify the notation we assume that the Lebesgue measure of the set D is 1. We start with part 1). Let δ_0 be a constant defined in Lemma 1.1. Note that

$$J_k(f) = \frac{1}{\alpha(X(k-1))}\int_D f(x)\beta_{n(x,X(k-1))}(x)dx, \qquad k \ge 1.$$

Fix an arbitrary $\varepsilon > 0$ and define

$$B_{k,\varepsilon} = \Big\{\sup_{x\in D} |\beta(x) - \beta_{n(x,X(k-1))}(x)| \ge \varepsilon\Big\}, \qquad k \ge 1. \tag{23}$$

One can write

$$\begin{aligned}\mathsf{E}\,|J_k(f) - J(f)|^p &= \mathsf{E}\,|J_k(f) - J(f)|^p\, I_{\{B_{k,\varepsilon}\}} + \mathsf{E}\,|J_k(f) - J(f)|^p\, I_{\{\overline{B}_{k,\varepsilon}\}}\\ &= S_1 + S_2,\end{aligned}$$

where by $I_{\{B\}}$ we denoted an indicator of an event B. It is easy to see that

$$\begin{aligned}J_k(f) - J(f) = &\ \frac{\int_D f(x)(\beta_{n(x,X(k-1))}(x) - \beta(x))dx}{\alpha(X(k-1))}\\ &+J(f)\frac{\int_D(\beta(x) - \beta_{n(x,X(k-1))}(x))dx}{\alpha(X(k-1))},\end{aligned} \tag{24}$$

hence

$$|J_k(f) - J(f)| \leq \frac{2\|f\|_\infty}{\beta_{\min}} \sup_{x \in D} \left|\beta(x) - \beta_{n(x,X(k-1))}(x)\right|. \tag{25}$$

Let $k(\varepsilon)$ be such that $\|\beta_k - \beta\|_\infty \leq \varepsilon$ for any $k > k(\varepsilon)$. Then for any $k > k(\varepsilon)$ we can bound

$$S_1 \leq C\varepsilon^p. \tag{26}$$

Using Lemma 1.1 we have that for sufficiently large k

$$S_2 \leq \left(\frac{4\|f\|_\infty \beta_{\max}}{\beta_{\min}}\right)^p \mathsf{P}\{\overline{B}_{k,\varepsilon}\} \leq Ce^{-\lambda k}. \tag{27}$$

Combining bounds (26) and (27) we get that for all sufficiently large k

$$\|J_k(f) - J(f)\|_{L^p}^p \leq C(\varepsilon^p + e^{-\lambda k}).$$

Therefore L^p-convergence of $J_k(f)$ to $J(f)$ is proved for any $p > 1$, since ε was taken arbitrary. Part 1) of the lemma is proved.

Let now the condition (5) holds. Fix an arbitrary $\delta \in (0, \delta_0)$ and define

$$B_{k,\delta} = \left\{ \inf_{x \in D} n(x, X(k-1)) \geq k\delta \right\},\ k \geq 1. \tag{28}$$

One can repeat the reasonings above using this sequence of events instead of the events (23) and get the bound $S_1 \leq C\varphi^p(k\delta)$, therefore part 2) of Lemma 1.2 is also proved. □

Remark 3. Note that in the Eq. (25) it is also possible to bound

$$|J_k(f) - J(f)| \leq \frac{2\|f\|_1}{\beta_{\min}} \sup_{x \in D} \left|\beta(x) - \beta_{n(x,X(k-1))}(x)\right|, \tag{29}$$

where $\|f\|_1 = \int_D |f(x)|dx$.

Proof of Corollary 1.2. By the binomial formula we have that

$$\left|\mathsf{E}\left(\xi_k^n(f) \mid \mathcal{F}_{k-1}\right) - \mathcal{U}_n(f)\right| \leq \sum_{i=0}^{n} \binom{n}{i} \left|J_k(f^i)\,(\mathsf{E} f(X_k))^{n-i} - J(f^i)\,J^{n-i}(f)\right|.$$

Noting that

$$\left|J_k(f^i)(\mathsf{E} f(X_k))^{n-i} - J(f^i)J^{n-i}(f)\right| \leq C\left(\left|J_k(f^i) - J(f^i)\right| + |\mathsf{E} f(X_k) - J(f)|\right)$$

and applying part 1) of Lemma 1.2 we prove part 1) of the corollary. If the condition (4) holds, then by part 2) of Lemma 1.2 we can bound for any $\delta \in (0, \delta_0)$

$$\mathsf{E}|J_k(f^i) - J(f^i)| + |\mathsf{E} f(X_k) - J(f)| \leq C(\varphi(k\delta) + e^{-\lambda k}) \tag{30}$$

and part 2) of the corollary is also proved. □

Proof of Lemma 1.3. We can write

$$\mathsf{E}\left(\eta \prod_{v=1}^{k} \xi_{i_v}^{r_v}(g_v)\right) = \mathcal{U}_{r_k}(g_k)\mathsf{E}\left(\eta \prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\right)$$
$$+ \mathsf{E}\left(\eta \prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\left(\mathsf{E}\left(\xi_{i_k}^{r_k}(g_k) \mid \mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_k)\right)\right).$$

The functions g's are bounded, so

$$\left|\mathsf{E}\eta \prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\left(\mathsf{E}\left(\xi_{i_k}^{r_k}(g_k) \mid \mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_k)\right)\right|$$
$$\leq C_1^{n-r_k}\mathsf{E}\left|\mathsf{E}\left(\xi_{i_k}^{r_k}(g_k) \mid \mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_k)\right|,$$

and the right hand side above goes to 0 as $i_k \to \infty$ by part 1) of Corollary 1.2. If the condition (5) holds, then by part 2) of Corollary 1.2 we can bound

$$\mathsf{E}|\mathsf{E}\left(\xi_{i_k}^{r_k}(g_k) \mid \mathcal{F}_{i_k-1}\right) - \mathcal{U}_{r_k}(g_k)| \leq C_2(\varphi(i_k\delta) + e^{-\lambda i_k})$$

for any $\delta \in (0, \delta_0)$ with some $\lambda = \lambda(\delta)$. Repeating the same arguments for the indices $i_{k-1}, \ldots, i_1$ in $\mathsf{E}\big(\eta \prod_{v=1}^{k-1} \xi_{i_v}^{r_v}(g_v)\big)$ we finish the proof. □

Proof of Lemma 1.4. Let us prove that

$$\frac{1}{\sqrt{m}} \sum_{k=1}^{m} (J_k(f) - J(f)) \to 0, \tag{31}$$

in probability as $m \to \infty$. Using the bound (25) we get that

$$|J_k(f) - J(f)| \leq C_1 \int_D \left|\beta_{n(x,X(k-1))}(x) - \beta(x)\right| dx\, I_{\{B_{k,\delta}\}} + C_2 I_{\{\overline{B}_{k,\delta}\}}$$

where $B_{k,\delta}$ is the event defined by the Eq. (28). Therefore

$$\frac{1}{\sqrt{m}}\left|\sum_{k=1}^{m}(J_k(f) - J(f))\right| \leq \frac{C_1}{\sqrt{m}} \sum_{k=1}^{m} \int_D \left|\beta_{n(x,X(k-1))}(x) - \beta(x)\right| dx\, I_{\{B_{k,\delta}\}}$$
$$+ \frac{C_2}{\sqrt{m}} \sum_{k=1}^{m} I_{\{\overline{B}_{k,\delta}\}}. \tag{32}$$

By Lemma 1.1

$$\sum_{k=1}^{\infty} \mathsf{P}\{\overline{B}_{k,\delta}\} < \infty,$$

hence by Borel-Cantelli lemma only a finite number of events $\overline{B}_{k,\delta}$ occurs with probability 1, so

$$\frac{C_2}{\sqrt{m}} \sum_{k=1}^{m} I_{\{\overline{B}_{k,\delta}\}} \to 0$$

almost surely as $m \to \infty$. The first sum in the right hand side of the Eq. (32) is bounded by

$$\frac{C_1}{\sqrt{m}} \sum_{k=1}^{m} \sup_{x \in D} \left| \beta_{n(x, X(k-1))}(x) - \beta(x) \right| I_{\{B_{k,\delta}\}} \leq \frac{C_3}{\sqrt{m}} \sum_{k=0}^{m} \varphi(k\delta),$$

and it goes to 0 as $m \to \infty$ because of the Eq. (5). Repeating the same arguments we can also prove that

$$\frac{1}{\sqrt{m}} \sum_{k=1}^{m} (\mathsf{E} f(X_k) - J(f)) \to 0,$$

as $m \to \infty$, therefore Lemma 1.4 is proved. □

3.. EXPONENTIAL RATE OF CONVERGENCE

If the rate of convergence in (4) is exponential, namely if $\varphi(k) = \exp(-\gamma k)$ for some $\gamma > 0$, then stronger statement of asymptotic independence of random variables X_k, $k \geq 1$, can be made. Fix some $0 < \varepsilon < 1/2$ and denote

$$\tilde{S}_m(f) = \frac{1}{\sqrt{m - m^{\varepsilon}}} \sum_{k=m^{\varepsilon}}^{m} (f(X_k) - \mathsf{E} f(X_k)).$$

Let Y_i, $i \geq 1$, be a collection of independent random variables with the common probability density $\beta(x)/\alpha$. Denote

$$S_{0,m}(f) = \frac{1}{\sqrt{m_{\varepsilon}}} \sum_{k=1}^{m_{\varepsilon}} (f(Y_k) - \mathsf{E} f(Y_k)),$$

where we denoted $m_{\varepsilon} = m - m^{\varepsilon}$. We are going to show that for a fixed set of positive indexes $r_1, \ldots, r_k$, such that $r_1 + \cdots + r_k = n$ the following expansion holds

$$\prod_{j=1}^{k} \mathsf{E}\tilde{S}_m^{r_j}(f) = \prod_{j=1}^{k} \mathsf{E}S_{0,m}^{r_j}(f) + \zeta_m(r_1, \dots, r_k, f), \tag{33}$$

where

$$|\zeta_m(r_1, \dots, r_k, f)| \le C(n) m^{\varepsilon + n/2} e^{-\rho m^{\varepsilon}}.$$

For the simplicity of notation we prove the expansion (33) for the particular case $k = 1, r_1 = n$. It is easy to see that

$$\mathsf{E}\tilde{S}_m^n(f) = m_\varepsilon^{-n/2} \sum_{t_1,\dots,t_p} \sum_{m_\varepsilon \le i_1 < \dots < i_p \le m} \mathsf{E} \prod_{v=1}^{p} \xi_{i_v}^{t_v}(f),$$

where the first sum is over all sets of positive integers $t_i, i = 1, \dots, p$, such that $t_1 + \dots + t_p = n$. We get the expansion (33) if we put

$$\zeta_m(n, f) = m_\varepsilon^{-n/2} \sum_{t_1,\dots,t_p} \sum_{m_\varepsilon \le i_1 < \dots < i_p \le m} \left(\mathsf{E} \prod_{v=1}^{p} \xi_{i_v}^{t_v}(f) - \prod_{v=1}^{p} \mathcal{U}_{t_v}(f) \right).$$

Applying the bound (10) with $\varphi(k) = \exp(-\gamma k)$ yields that

$$\left| \mathsf{E} \prod_{v=1}^{p} \xi_{i_v}^{t_v}(f) - \mathsf{E} \prod_{v=1}^{p} \mathcal{U}_{t_v}(f) \right| \le C \sum_{v=1}^{p} e^{-\rho i_v},$$

where $\rho = \min(\gamma, \lambda)$. Therefore we get that

$$|\zeta_m(n, f)| \le m_\varepsilon^{-n/2} \sum_{t_1,\dots,t_p} \sum_{m_\varepsilon \le i_1 < \dots < i_p \le m} C \sum_{v=1}^{p} e^{-\rho i_v}. \tag{34}$$

It is easy to see that for any fixed set of positive integers $t_1, \dots, t_p$ in the first sum we can bound

$$\begin{aligned} m_\varepsilon^{-n/2} \sum_{m_\varepsilon \le i_1 < \dots < i_p \le m} C_1^n \sum_{v=1}^{p} e^{-\rho i_v} &\le C_2 m^{(\varepsilon - 1/2)n} (m - m^\varepsilon)^{p-1} e^{-\rho m^\varepsilon} \\ &\le C_3 m^{\varepsilon + n/2} e^{-\rho m^\varepsilon}. \end{aligned}$$

The first sum in (34) contains the number of terms depending only on n, therefore

$$|\zeta_m(n, f)| \le C_4 m^{\varepsilon + n/2} e^{-\rho m^\varepsilon}.$$

Using the representation (33) we can prove that $\mathcal{K}_{mn}(f)$ the nth cumulant of $\tilde{S}_m(f)$ converges as $m \to \infty$ to the cumulant of a Gaussian random variable with zero mean and the variance $G(f, f)$. Using Lemma 1.3 it is easy to prove that $\mathcal{K}_{m2}(f) \to G(f, f)$ as $m \to \infty$. Let us prove that $\mathcal{K}_{mn}(f) \to 0$ as $m \to \infty$

for $n \geq 3$. Recall that the cumulants $\mathcal{K}_{mn}(f), n \geq 1$, are defined as the Taylor coefficients of the logarithm of the characteristic function

$$\log \mathsf{E}e^{it\tilde{S}_m(f)} = \sum_{n=1}^{\infty} \mathcal{K}_{mn}(f)\frac{(it)^n}{n!}, \qquad t \in \mathbf{R}. \tag{35}$$

Each cumulant can be presented as a finite linear combination of the products of moments (see, for instance, Ref. 4)

$$\mathcal{K}_{mn}(f) = \sum_{k=1}^{n}(-1)^{k-1}(k-1)! \sum_{r_1,\ldots,r_k} \prod_{j=1}^{k} \mathsf{E}\tilde{S}_m^{r_j}(f), \tag{36}$$

where the second sum is over all sets of positive integers $\{r_1, \ldots, r_k\}$ such that $r_1 + \cdots + r_k = n$. The Eq. (33) yields that

$$\mathcal{K}_{mn}(f) = \mathcal{K}_{mn}^{(0)}(f) + \sum_{k=1}^{n}(-1)^{k-1}(k-1)! \sum_{r_1,\ldots,r_k} \zeta_m(r_1, \ldots, r_k, f), \tag{37}$$

where $\mathcal{K}_{mn}^{(0)}(f)$ is nth cumulant of the random variable $S_{0,m}(f)$. Because of independence we have that $\mathcal{K}_{mn}^{(0)}(f) \sim m_\varepsilon^{-n/2+1} \to 0$ for any $n > 2$ as $m \to \infty$. It remains to note that

$$\left|\sum_{k=1}^{n}(-1)^{k-1}(k-1)! \sum_{r_1,\ldots,r_k} \zeta_m(r_1, \ldots, r_k, f)\right| \leq C_5 n! m^{\varepsilon+n/2} e^{-\rho m^\varepsilon} \to 0,$$

as $m \to \infty$. Thus the convergence of cumulants is proved. It is well known that this implies the weak convergence.

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